Effect of Non-Normality Dynamics on the Expected Return of Options

ILYA FIGELMAN

ILYA FIGELMAN
is a vice president and quantitative analyst at AllianceBernstein in New York, NY.
ilya.figelman@alliancebernstein.com

In Figelman [2008], I provided a framework for determining the real world expected return and risk of options. That framework is used to compare the efficacy of stock index at-the-money covered call (CC) strategies relative to the stock index itself.[1] The strong relative historical performance of a CC stock index strategy is theoretically explained by the framework. However, that framework assumes that the instantaneous stock index return is normally distributed; that is, the stock index process follows geometric Brownian motion.[2] In reality, it is well known that stock prices exhibit other stochastic dynamics, such as jumps and stochastic volatility.

In this article, we will analyze how the real-world expected return of options, and thus CC strategies, is affected when stock prices follow a process that is more realistic than geometric Brownian motion. Via a Monte Carlo simulation, such a stock index process is generated, which allows us to measure the expected return of options and the profitability of CC strategies. This profitability is compared to the one obtained when stock index returns are assumed to follow standard geometric Brownian motion. Somewhat surprisingly, our results show that the real-world expected value of an at-the-money call option does not increase when stock index returns are assumed to follow a more realistic process.[3] In addition, the risk of the CC strategy relative to the stock index does not materially change when a more realistic process is introduced.[4,5] Thus, the fact that instantaneous stock returns are not normally distributed does not detract from the efficacy of CC strategies (relative to the stock index) that employ at-the-money call options. This result also implies that non-normality dynamics of the stock index cannot explain the positive historical at-the-money implied versus realized volatility spread thoroughly discussed in Figelman [2008].

ERAKER, JOHANNES, AND POLSON'S STOCHASTIC VOLATILITY WITH CORRELATED JUMPS MODEL (SVCJ)

For a more realistic stock return process, we employ the Eraker, Johannes, and Polson (EJP) [2003] stochastic volatility with correlated jumps (SVCJ) model.[6] As its name suggests, the SVCJ model incorporates 1) standard geometric Brownian motion of returns; 2) stochastic volatility, or geometric Brownian motion of volatility that is (negatively) correlated with returns; 3) jumps in returns; and 4) jumps in volatility that are (positively) correlated with jumps in returns.

EJP [2003] provided both continuous and discrete specifications of the SVCJ model; we will describe both here. The equations for the continuous EJP SVCJ model are

$$dY_t = \mu_{\ln} dt + \sqrt{V_t dt} dW_{t,1} + \xi^y dN_t \tag{1}$$

$$dV_t = \kappa(\theta - V_t)dt + \sqrt{V_t dt}\sigma_v(\rho dW_{t,1} + (1-\rho)dW_{t,2}) + \xi^v dN_t \tag{2}$$

$$\xi^v \sim \exp(\mu_v) \tag{3}$$

$$\xi^y \mid \xi^v \sim \Phi(\mu_y + \rho_J \xi^v, \sigma_y^2) \tag{4}$$

where

- Y_t = $\ln(S_t/S_0)$;
- V_t = variance of Y_t;
- dW = standard Brownian motion;
- dN = Poisson process;
- exp = exponential distribution; and
- Φ = normal distribution.

Notice that if $V_t = \theta$ for all t and $\xi_y = 0$, then Equation (1) reduces to the standard geometric Brownian motion, as shown in Figelman [2008, p. 82, Eq. (3)] with[7]

$$\mu_{\ln} = \mu - \sigma_f^2/2 \tag{5}$$

and

$$\theta = \sigma_f^2 \tag{6}$$

As previously mentioned, the SVCJ model contains three stochastic dynamic effects in addition to geometric Brownian motion.

Stochastic Volatility

EJP [2003] incorporated the Heston [1993] square-root stochastic volatility model, which is a simplified version of Equations (1) and (2), when $\xi^y = \xi^v = 0$. The intuition behind this model is that stock index volatility is itself a random variable that is persistent, but reverts to a mean of θ with a rate κ. In addition, changes in volatility have a (negative) correlation ($\rho < 0$) with stock index returns.

Jumps in Returns

EJP [2003] also incorporated large, but rare, jumps in stock index returns to capture the crash dynamics of the stock market, such as those that occurred in October 1987 and November 1998. (In contrast, geometric Brownian motion captures small, but frequent, movements in stock returns.) The jump size, ξ^y, is normally distributed with a (negative) mean ($\mu_y < 0$) and a large standard deviation, σ_y. The jump times occur infrequently via a Poisson process, dN_t. Bates [1996] introduced a similar model.

Jumps in Volatility Correlated with Jumps in Returns

Another feature of the EJP SVCJ model is jumps in volatility, which are correlated with jumps in returns. The intuition is that when a large move in returns occurs, there is a contemporaneous large increase in volatility. In Equations (1) and (2), the jump times for returns and volatility are identical, dN_t, and the jump sizes in Equations (3) and (4), ξ^y and ξ^v, are (positively) correlated ($\rho_J > 0$). In addition, after a large move in returns, the market remains uncertain for a period of time, which causes volatility to remain high, but eventually revert to its mean. This dynamic occurred after the market crashes of October 1987 and November 1998 and is nicely captured by the persistent, yet mean-reversion nature of volatility shown in Equation (2). This SVCJ model was first introduced in Duffie, Pan, and Singleton (DPS) [2000]; DPS [2000] also introduced another model with jumps in volatility uncorrelated with jumps in returns. EJP [2003], however, convincingly argued that it is appropriate to assume a correlation between jumps in returns and volatility.

EJP [2003] estimated the parameters for a discrete specification of the SCVJ model as follows:

$$Y_{t+1} - Y_t = \mu_{\ln}\Delta t + \sqrt{V_t \Delta t}\varepsilon_{t+1}^y + \xi_{t+1}^y J_{t+1} \tag{7}$$

$$V_{t+1} - V_t = \kappa(\theta - V_t)\Delta t + \sigma_v\sqrt{V_t \Delta t}\varepsilon_{t+1}^v + \xi_{t+1}^v J_{t+1} \tag{8}$$

where J is a random variable that can be either 0 or 1, and whose mean is $\lambda\Delta t$; ε^y and ε^v are standard normal random variables with correlation ρ and Δt is one day.

Exhibit 1 shows the parameters estimated by EJP [2003].[8] EJP estimated these parameters using historical data from 1980 to 1999, which includes the market crashes of October 1987 and November 1998.

EFFECT OF NON-NORMAL STOCHASTIC DYNAMICS ON CC STRATEGY

Now, we will compare the EJP [2003] process to geometric Brownian motion in order to determine what effect more realistic stochastic dynamics have on the expected return of options, and thus the profitability of the CC strategy. For this comparison to be accurate, the stock index price expected return and realized volatility must be identical in the two processes.

For the stock index price expected returns to be equal under both processes, parameter $\mu_{\ln}$ needs to be adjusted by the effect from jumps in returns (see Equations (3) and (4)),

$$\mu_{\ln}^{BM} = \mu_{\ln}^{SVCJ} - E[\xi^y J] = \mu_{\ln}^{SVCJ} - \lambda(\mu_y + \rho_J \mu_v) \qquad (9)$$

For the stock index price total variances to be equal under both processes, parameter θ needs to be adjusted by the effects of jumps in returns, jumps in volatility, and stochastic volatility,

$$\theta^{BM} = \theta^{SVCJ} + (\mu_v \lambda)/\kappa + \mu_y^2 + 2\mu_y \mu_v \rho_J + \rho_J^2 \mu_v^2 + \sigma_y^2 \qquad (10)$$

This relationship is described in EJP [2003, p. 1273].

Exhibit 2 provides characteristics of the geometric Brownian motion stock index process.[9] As previously mentioned, we forced the total stock index expected return and total volatility under geometric Brownian motion to be the same as under the SVCJ process. The risk-free rate and dividend yield are just the historical averages for the same historical period (1980–1999) that EJP [2003] used to estimate their parameters. The at-the-money implied–realized volatility spread is assumed to be 2.0%.[10]

Exhibit 3 shows the profitability of the CC strategy under the two stock index processes calculated via a Monte Carlo simulation.[11] Somewhat surprisingly, the expected return of the CC strategy is not worse under the SVCJ stock index process than under geometric Brownian motion, and, in fact, it is actually slightly better (1.30% versus 1.26%, monthly, and 16.8% versus 16.2%, annualized). This occurs because in the EJP SVCJ model jumps in returns tend to be negative ($\mu_y < 0$), which causes the expected value of the at-the-money call option used in our CC strategy to be slightly less valuable under the SVCJ model than under geometric Brownian motion. Therefore, because the CC strategy is short the at-the-money call options, it is slightly more profitable under the SVCJ model than under geometric Brownian

EXHIBIT 1
Parameters in the EJP SVCJ Model

Parameter	The EJP SVCJ Model	Geometric Brownian Motion	Description
μ_{ln}	0.06925	0.08970	Mean return (w/o jumps)
Θ	0.5376	1.0524	Mean variance (w/o jumps)
K	0.0260		Variance reversion parameter
σ_y	0.0790		Volatility of return jumps conditional on volatility jumps
μ_y	−1.7533		Mean of return jumps conditional on volatility jumps
ρ_J	−0.6008		Correlation between return and volatility jumps
σ_v	2.8864		Volatility of variance (w/o jumps)
μ_v	1.4832		Mean of volatility jumps

Note: The parameter estimation period is from 1980 to 1999. Units of the parameters are percent daily values. To obtain annualized values, the return parameters need to be multiplied by 252/100 and variance parameters need to be multiplied by 252/10000. EJP [2003] called our parameter μ_{ln} just μ; Equation (5) relates our μ to our μ_{ln}, or μ in EJP [2003].

EXHIBIT 2

Input Characteristics of the EJP SVCJ and Geometric Brownian Motion Models

Time Horizon (t)	1 Month
Strike Price (K)	100.42%
Risk-Free Rate (r)	7.9%
Stock Index Price Expected Return (μ)	14.3%
Stock Index Dividend Yield (q)	2.8%
Equity Risk Premium	9.3%
Stock Index Price Implied Volality (σ_i)	18.0%
Stock Index Price Volality Spread ($\sigma_i - \sigma_f$)	2.0%
Stock Index Price Future Realized Volatility (σ_f)	16.0%

Note: Exhibit 2 transforms some of the parameters in Exhibit 1 to be consistent with the format used in Figelman [2008].

EXHIBIT 3

CC and Stock Index Expected Return and Risk: EJP SVCJ Model vs. Geometric Brownian Motion

	Geometric Brownian Motion	EJPs SVCJ Model
Expected Return (Annualized)		
Stock Index	18.7%	18.7%
Covered Call	16.2%	16.8%
Expected Return (Monthly)		
Stock Index	1.44%	1.44%
Covered Call	1.26%	1.30%
Semi-standard Deviation (Monthly)		
Stock Index	2.82%	3.00%
Covered Call	1.88%	2.16%
Sharpe Ratio with Semi-standard Deviation (Monthly)		
Stock Index	0.28	0.26
Covered Call	0.32	0.30
Real World Implied Volatility (Annualized)	16.0%	15.7%

motion. As we will soon show, this is not the case for CC strategies implemented with away-from-the-money call options.

The CC strategy semi-standard deviation[12] is higher under the SVCJ model than under geometric Brownian motion (2.16% versus 1.88%, monthly) leading to a slightly lower Sharpe ratio with semi-standard deviation under the SVCJ model (0.30 versus 0.32, monthly). However, the stock index semi-standard deviation is also higher under the SVCJ model than under geometric Brownian motion (3.00% versus 2.82%, monthly) and yields a slightly lower Sharpe ratio with semi-standard deviation for the stock index under the SVCJ model (0.26 versus 0.28, monthly). By design, the expected return for the stock index is identical under both processes (1.44%, monthly, and 18.7%, annualized). The CC strategy monthly Sharpe ratio with semi-standard deviation is higher than that of the stock index under both processes. In addition, the difference in the monthly Sharpe ratios with semi-standard deviation between the processes is very similar for the CC strategy and the stock index. Given this evidence, it would be fair to say that jumps in returns and other stochastic dynamics in the stock index process do not change the profitability of the CC strategy in any significant manner. Thus, the framework presented in Figelman [2008] is a valid proxy for calculating the expected returns of at-the-money options and, thus, for evaluating at-the-money CC strategies.

REAL-WORLD IMPLIED VOLATILITY

Because it is intuitive to express option values in terms of implied volatility, we define a new volatility metric, shown in Exhibit 3, called real-world implied volatility. This volatility metric is implied from the real-world expected value of the call option (calculated via a Monte Carlo simulation) by using the inverse of the Rubinstein [1984] (real-world) version of the Black–Scholes model given in Figelman [2008, p. 83, Eq. (7)],

$$\sigma_{rwi} = BSINV(S_0 e^{\mu t}, K, E[C_t], 0, 0, t) \tag{11}$$

where *BSINV* is the inverse of the standard Black–Scholes formula. The corresponding market implied volatility is

$$\sigma_i = BSINV(S_0, K, C_0, r, q, t) \tag{12}$$

The real-world implied volatility under the SVCJ process is obtained by first calculating the expected real-world call option value ($E[C_t]$) via a Monte Carlo simulation and then plugging this value into Equation (11). The real-world implied volatility under geometric Brownian motion is identical to the expected forward-realized volatility, since the Black–Scholes formula assumes geometric Brownian motion. Therefore, real-world implied volatility can be thought of as what the expected forward-realized volatility needs to be for the expected call option value ($E[C_t]$) under real-world geometric Brownian motion to be equal to the expected call option value under another real-world process (e.g., SVCJ).[13] Exhibit 4 compares the properties of various volatility metrics.

Examining the real-world implied volatility is another way to measure the effect that the EJP SVCJ stock index process has on the expected value of a call option and, thus, on the CC strategy. Exhibit 3 shows that the call's real-world implied volatility (15.7%) under the SVCJ process is slightly lower than that (16.0%) under geometric Brownian motion. This is consistent with the CC strategy expected return being slightly higher under the SVCJ process than under geometric Brownian motion.

Market implied volatility, shown in Equation (12), is what the forward-realized volatility needs to be for a call option present value under a risk-neutral measure (assuming a stock index geometric Brownian motion process) to be equal to the call option price. The call option price is determined by the market maker's view of the risk-neutral process as well as by other factors, such as option supply and demand. Market makers use a risk-neutral stock index process because they are able to dynamically hedge with the stock index. The CC strategy is buy and hold, with no dynamic hedging, so that investors use the real-world stock index process instead. If market makers and investors assume the same stock index dynamics, the only difference between the risk-neutral and real-world processes is the stock index expected return; the stock index realized volatility and other stochastic dynamics are the same (see Hull [2003, Ch. 12]). Therefore, if investors and market makers assume the same stock index price dynamics, then the difference between the real-world and market implied volatilities cannot be due to these dynamics and must be due to other factors, such as option supply and demand.

Figelman [2008] showed that the historical at-the-money market implied volatility for the S&P 500 has been higher, on average, than the forward realized volatility (by approximately 2.00%). Theoretically, it could be possible for this spread to be due to the fact that stock index returns follow a different stochastic process than the geometric Brownian motion assumed in the Black–Scholes formula. In other words, market makers may be incorporating non-normality dynamics in their pricing models, which may cause the at-the-money implied volatility to be higher, on average, than the forward realized volatility. If this were true, then the real-world implied volatility would also be higher than the average forward-realized volatility, because, as just explained, the stochastic dynamics affect the implied volatilities under the real-world and risk-neutral processes in the same manner. However, the analysis in this section shows that non-normal stochastic dynamics do not cause at-the-money real-world implied volatility to be higher than the forward realized volatility. Exhibit 3 shows that the real-world implied volatility under the SVCJ process is actually slightly lower than that under geometric Brownian motion, the latter being equivalent to the forward realized volatility. Thus, the market implied volatility under the SVCJ risk-neutral process would also need to be slightly lower than the forward realized volatility. Therefore, the fact that at-the-money market implied volatility actually tends to be higher, on average, than the forward realized volatility cannot be due to the non-normality dynamics of the stock index process.

EXHIBIT 4

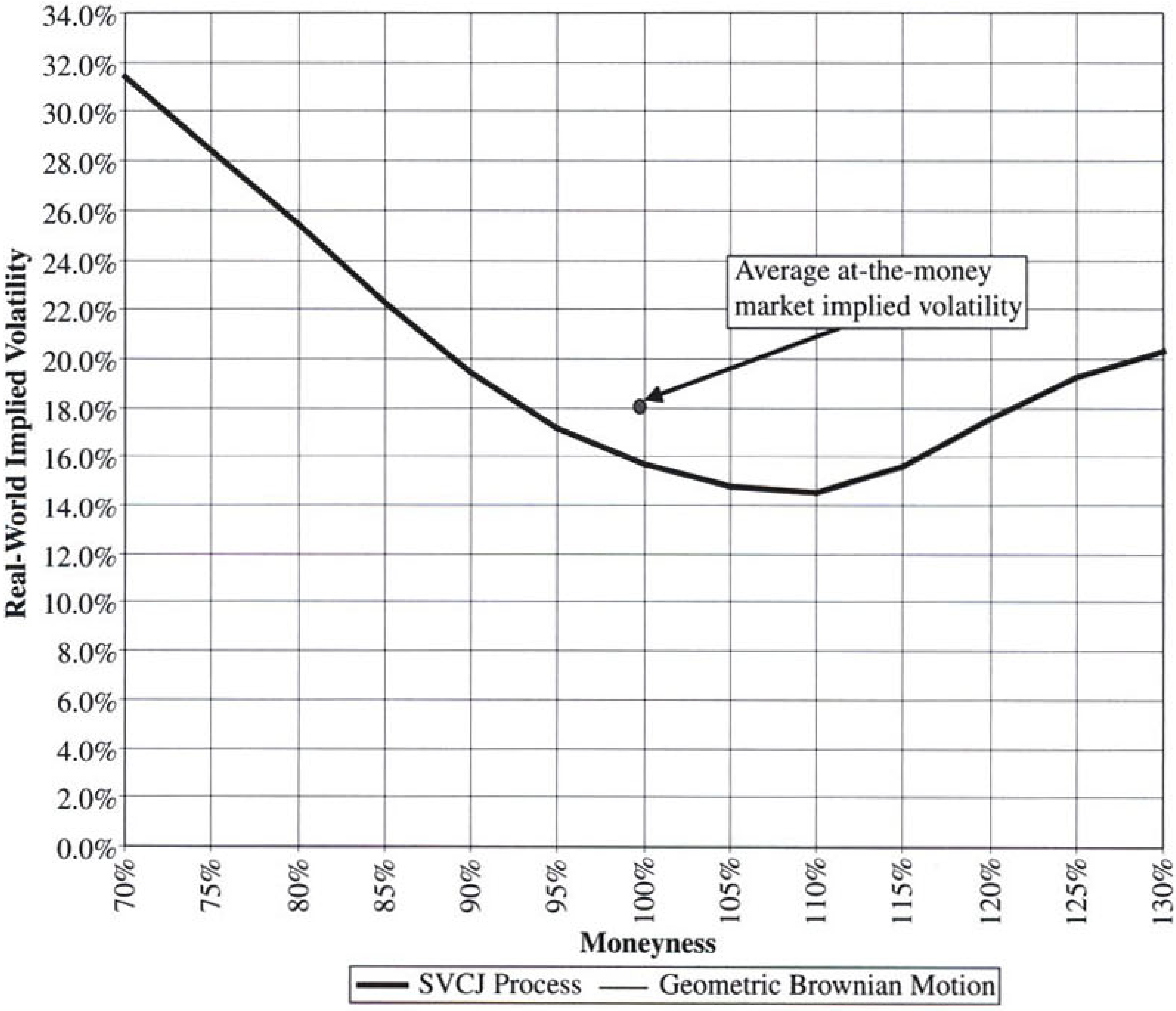
Characteristics of Volatility Metrics

	Expected Realized Volatility	Real-World Implied Volatility	Market Implied Volatility
Affected by stock price dynamics (for a given level of realized volatility)	No	Yes	Yes
Affected by option market supply/demand	No	No	Yes
Depends on investor-specific forecast(s)	Yes	Yes	No
Depends on option strike	No	Yes	Yes

Basically, all else being equal, the existence of jumps and other stochastic dynamics does not cause at-the-money options to be more valuable and thus does not cause a higher real-world market implied volatility.[14] Instead, it is more plausible that the spread between at-the-money market implied volatility and forward realized volatility is mostly due to the demand for options, as argued by Bollen and Whaley [2004] and Garleanu, Pedersen, and Poteshman [2005].

We also calculated the real-world implied volatilities for the SVCJ stock index process for in- and out-of-the-money call options. Exhibit 5 shows a volatility skew/smile in the real-world implied volatility of the SVCJ process, which is also typical for market implied volatility. The shape of the real-world implied volatility curve is driven by the dynamics of the jumps in the stock index returns of the SVCJ model, although stochastic volatility and jumps in volatility contribute to the shape as well. First, the value of away-from-the-money calls is positively related to the magnitude of jump sizes (i.e., the deeper the call is away from the money, the stronger the relationship). Second, the negative mean of the jumps ($\mu_y < 0$, see Exhibit 1) causes the value of in-the-money calls to increase and out-of-the-money calls to decrease (i.e., the closer the call is to being at-the-money, the stronger the relationship). These two dynamics cause the in-the-money call options to be significantly more valuable (in expectation) under the SVCJ process than under geometric Brownian motion. These two dynamics also cause near out-of-the-money call options to be less valuable and deep out-of-the-money call options to be more valuable under SVCJ relative to geometric Brownian motion.[15] These two effects cause this skew/smile shape of the curve in Exhibit 5.

EXHIBIT 5
Real-World Implied Volatility



Note: Exhibits 1 and 2 show parameters of the process and characteristics of the call option.

The market implied volatility curve would differ from the real-world implied volatility curve. Market implied volatilities also incorporate other factors, such as option supply and demand. Bondarenko [2003], Bollen and Whaley [2004], and Garleanu, Pedersen, and Poteshman [2005] evaluated the profitability of strategies that systematically short away-from-the-money options and found that they are even more profitable than strategies that systematically short at-the-money options. This suggests that market implied volatility is also significantly higher than real-world implied volatility for away-from-the-money options. This difference is also probably due to option demand and not to non-normality stochastic dynamics. We leave the comprehensive evaluation of away-from-the-money options, in the context of the framework presented here, for further research.

ENDNOTES

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[1]Comparing a CC strategy relative to the stock index is a convenient way to measure the efficacy of systematically selling options. Simply selling naked options causes implicit leverage, which makes it challenging to measure returns in a traditional sense. A CC is a fully invested (unlevered) strategy, whose returns can be easily measured; this is one reason why the CBOE has a CC S&P 500 Index, BXM. However, one needs to be aware that relative to the stock index, a CC strategy trades volatility exposure in exchange for stock market exposure. See the introduction of Figelman [2008] for more details.

[2]As shown in Figelman [2008] on p. 82, Equation (3).

[3]This is obviously not the case for call options away from the money.

[4]In this article, risk is defined as semi-standard deviation. Appendix B in Figelman [2008] discusses the pros and cons of this measure.

[5]The risk of both a CC strategy and the stock index itself is higher under the more realistic process; however, the difference between the risks of these strategies is very similar under both processes.

[6]The literature provides other models similar to the EJP [2003] SVCJ model, such as those in Duffie, Pan, and Singleton [2000], Pan [2002], Eraker [2004], and Broadie, Chernov, and Johannes [2005]. The EJP model, however, is the only one that solely uses stock returns to estimate its parameters. These other models mix option price information with stock returns. We only use the EJP SVCJ model to calculate the real-world expected value of a call option ($E[C_t]$), as given in Figelman [2008, p. 83, Eq. (7)], under a more realistic real-world stock return process than geometric Brownian motion. Investors in CC strategies are assumed to be price takers, thus, the price of the call option (C_0), as given in Figelman [2008, p. 82, Eq. (6)], is just obtained from the market via an implied volatility. Therefore, a model that mixes stock index returns and option prices would not suit our purposes.

[7]See Hull [2003, pp. 227–228] for a detailed description of the relationship in Equation (5).

[8]These parameters are also shown in Table III (p. 1280) of EJP [2003]. We made one adjustment to the EJP parameters. We increased $\mu_{\ln}$ such that the expected value of the total log stock index price return is equal to its historical value (13.05%) for the time period EJP used in their estimation (1980–1999). The EJP estimation method does not force this result and yields a much lower value (9.56%). We made this adjustment in order that our Monte Carlo results would make sense from an economic perspective. This adjustment, however, does not affect our conclusions about expected value of options using the EJP process versus geometric Brownian motion, because we forced the expected return and volatility to be the same under both processes. The stock index price total volatility from the EJP model (15.99%) is very close to the historical value (15.88%); therefore, we do not adjust their volatility parameters.

[9]Equations (5) and (6) relate μ to $\mu_{\ln}$ and σ_f to θ for geometric Brownian motion. Equations (9) and (10) relate the parameters of the SVCJ process to those of geometric Brownian motion. All of the parameters of the SVCJ process are provided in Exhibit 1.

[10]In Figelman [2008, pp. 86–88], the average implied–realized volatility spread (for an investor) was shown to average approximately 2.0% between 1994 and 2005. Unfortunately, good option market prices and implied volatilities are not available prior to the crash of 1987. Thus, this parameter is estimated on a different time period than the EJP SVCJ model (1980–1999). The goal of this analysis, however, is to compare the relative efficacy of a CC strategy under two real-world processes, which should not be affected by assumed implied volatility. Only the absolute efficacy of the CC strategy is affected by this assumption.

[11]To ensure accuracy of the results, one million paths were used, which is an order of magnitude greater than similar analyses in the literature.

[12]See Figelman [2008] Appendix B for the pros and cons of semi-standard deviation as a risk measure.

[13]Real-world implied volatility is the "wrong" volatility metric plugged into the "wrong" model (Black–Scholes) to obtain the "right" option expected value. (A similar concept is described by Peter Carr in many of his talks.)

[14]This is obviously not the case for away-from-the-money options.

[15]Due to put–call parity, the real-world implied volatility of out-of-the-money puts is equivalent to in-the-money calls, and the vise versa.

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